RADIANT HEAT TRANSFER OF A NONGRAY GAS WITH GRAY SURFACES

S. P. Detkov

UDC 536.248:535.211

In practical calculations of radiant heat transfer the chamber volume is regarded as a single zone with a uniform effective temperature. The walls are gray. Exact and approximate methods of calculation for such a model are given.

§1. Introduction

In a system of gray surfaces bounding a transparent gas, radiation undergoes multiple reflections before being absorbed by the sink; an accurate treatment of this situation is very difficult. Hottel [1] gave a general approximate solution of the problem of radiant heat transfer. He suggested a selectively gray model of the gas spectrum. The zone equations are similar to those obtained for a gray gas and this is their advantage. The only exact solution is that given by Hottel and Egbert [2] for a gray enclosure containing a gas. This solution involves effective beam lengths and functions $\varepsilon(x, T)$ and $a(x, T)$. In [3] the solution was simplified a little by regrouping of the terms. Below we obtain solutions in the same sense, accurate for systems: 1) with concave gray and black surfaces; 2) with two nonconcave gray surfaces and two concave black surfaces. An approximate solution based on curtailment of the series giving the exact solution is proposed.

§2. General Solution for Any Number of Zones

A system of gray volume and surface zones is a suitable standard giving the initial equations. For an arbitrary number of zones with isotropic reflection and scattering, according to [4],

$$Q_{ab} = A\tilde{\Phi}Q_0.$$

Here Q_{ab} and Q_0 are the columns of absorbed and original fluxes; A is the diagonal matrix of absorption coefficients; $\tilde{\Phi}$ is the transposed square matrix of "resolving" angular coefficients [4, 5]. It can be determined from the inverse matrix of the zone equation coefficients

$$I + \tilde{\Phi}R = (I - \tilde{\varphi}_* R)^{-1}.$$

Here I is the unit matrix; R is the diagonal matrix of the reflection or scattering coefficients; $\tilde{\varphi}^*$ is the transposed square matrix of angular coefficients. In [4] a numerical example of determination of the matrix Φ was given.

The column of resultant fluxes has the form

$$Q_r = Q_{ab} - Q_0 = (I - A\tilde{\Phi})\, Q_0. \tag{1}$$

In the usual notation

$$Q_{ri} = \sum_{k=1}^{n} A_i \Phi_{ki} Q_{0k} - Q_{ci}$$

or

$$Q_{ri} = \sum_{k=1}^{n} Q_{0k} N_{ki} - Q_{0i} \quad (i = 1,\ 2,\ \ldots,\ n). \tag{2}$$

Ural Electromechanical Institute of Railroad Engineers, Sverdlovsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 18, No. 1, pp. 5-11, January, 1970. Original article submitted March 10, 1969.

Here $N_{ki} = \Phi_{ki}A_i$ is the dimensionless "resolving" angular coefficient, being the probability that a photon emitted by zone k will arrive in zone i and be absorbed by zone i.

We can apply equations (1) and (2) as approximations to a system with nongray zones if we use average angular coefficients which are independent of the number of reflections or scatterings. An attempt to determine such coefficients was made in [6]. In our case a single volume zone without scattering can be characterized (according to Hottel) by its emissivity, and the volume emission (absorption) can thus be replaced by an equivalent surface emission (absorption). The entire calculation is then based on known emissivity nomograms. In this case equations (1) or (2) give exact solutions if the resolving angular coefficients are calculated with all the multiple reflections taken into account. For the simplest systems the calculation procedure is shown below. In any case, it follows from the energy balance that

$$\sum_{i=1}^{n} N_{ki} = 1. \tag{3}$$

According to the reciprocity relation for $T_i = T_k$

$$F_i\Phi_{ik} = F_k\Phi_{ki} \quad \text{or} \quad A_iF_iN_{ik} = A_kF_kN_{ki}. \tag{4}$$

For the zone of the i-boundary of the volume with zone k, $A_i \equiv \varepsilon_i^0$; $F_i = F_k$.

§ 3. Angular Coefficients for Multiple Reflections

The angular coefficient for surfaces i and k, separated by a nongray medium, in the case of isotropic radiation has the form

$$\psi_{ik} = \frac{1}{F_i}\int_{F_i} dF_i \int_{F_k} D(l_{ik})\, l_{ik}^{-2}\cos\theta_i\cos\theta_k dF_k.$$

For a transparent medium $D = 1$, $\psi \equiv \varphi$. The use of the effective beam length (l_e) is equivalent to taking the transmittance $\psi = D(l_e)\varphi$ outside the integral. Consequently, the dependence of ψ on the number of reflections is manifested only through the function $D(l_e)$. As an example we consider a sequence of surfaces 1, 2, . . . , n. The probability that a photon emitted by surface 1 lands on surface n by successive reflections from surfaces 2, 3, . . . , n − 1, is

$$W_{1n} = \psi_{12}R_2\psi_{23}R_3 \ldots \psi_{n-2,n-1}R_{n-1}\psi_{n-1,n}$$
$$= \varphi_{12}\varphi_{23} \ldots \varphi_{n-1,n}D_1D_{2-1} \ldots D_{n-(n-1)}\prod_{i=2}^{n-1} R_i.$$

Here, for instance, D_{3-2} is the transmittance of the region between surfaces 2 and 3 for a beam with a spectrum at the start of this region. The beam is initially emitted by surface 1. Obviously,

$$D_1D_{2-1} \ldots D_{n-(n-1)} = D_n,$$

where D_n is the transmittance of the whole broken path for a beam with the initial spectrum. Hence,

$$W_{1n} = \varphi_{12}\varphi_{23} \ldots \varphi_{n-1,n}D_n\prod_{i=2}^{n-1} R_i.$$

The derived formula is valid for any sequence of surfaces. For instance, for t reflections of a surface onto itself

$$W = D_{t+1}\varphi_{ii}^{t+1}R_i^t.$$

Here D_{t+1} (denoted by D_m in the general case) is the transmittance of the broken path, consisting of (t + 1) straight portions. For a black beam $D_m = 1 - a_m$, where a_m is the absorptivity of the same broken path; $a_m = \varepsilon_m$ when the temperatures of the beam and gas are equal.

§ 4. Exact Formulas for Enclosure of Concave Surfaces – a Gray Surface i and a Black Surface k

Successive consideration of multiple reflections leads to the formulas:

$$N_{ii} = \varphi_{ii}A_i[1 - a_1 + \varphi_{ii}R_i(1 - a_2) + \varphi_{ii}^2R_i^2(1 - a_3) + \ldots],$$

$$N_{ik} = \varphi_{ik}[1 - a_1' + \varphi_{ii}R_i(1 - a_2') + \varphi_{ii}^2 R_i^2 (1 - a_3') + \ldots],$$
$$N_{ig} = \varphi_{ii}A_i[a_1 + \varphi_{ii}R_i a_2 + \ldots] + \varphi_{ik}[a_1' + \varphi_{ii}R_i a_2' + \ldots].$$

The formulas for N_{ii} and N_{ik} can be curtailed

$$N_{ii} = \frac{\varphi_{ii}A_i}{1 - R_i\varphi_{ii}} - \varphi_{ii}A_i(a_1 + \varphi_{ii}R_i a_2 + \varphi_{ii}^2 R_i^2 a_3 + \ldots),$$
$$N_{ik} = \frac{\varphi_{ik}}{1 - R_i\varphi_{ii}} - \varphi_{ik}(a_1' + \varphi_{ii}R_i a_2' + \varphi_{ii}^2 R_i^2 a_3' + \ldots).$$

The dashes attached to a_m here and henceforth denote their differences as regards broken effective beam lengths. The broken beam path is easily determined from the series of angular coefficients φ after removal of the parentheses. Then:

$$N_{kk} = \varphi_{kk}(1 - a_1'') + \varphi_{ki}R_i\varphi_{ik}[(1 - \varphi_{ii}R_i)^{-1} - (a_2^* + \varphi_{ii}R_i a_3^* + \ldots)],$$
$$N_{ki} = \varphi_{ki}A_i[(1 - \varphi_{ii}R_i)^{-1} - (a_1' + R_i\varphi_{ii}a_2' + \varphi_{ii}^2 R_i^2 a_3' + \ldots)],$$
$$N_{kg} = \varphi_{kk}a_1'' + \varphi_{ki}R_i\varphi_{ik}(a_2^* + \varphi_{ii}R_i a_3^* + \ldots) + \varphi_{ki}A_i(a_1' + \varphi_{ii}R_i a_2' + \ldots).$$

It is easy to see that equation (3) is fulfilled. It served as a control for the formulas. The values of N_{gk} and N_{gi} can be determined from relationship (4), which in the given case has the form

$$N_{kg} = \varepsilon_{gk}N_{gk}, \quad A_i N_{ig} = \varepsilon_{gi}N_{gi}.$$

In the formulas for N_{kg} and N_{ig} we make the substitution $a_m \to \varepsilon_m$. ε_{gi} and ε_{gk} are the volume emissivities for surfaces i and k, determining the original fluxes from the volume to the surfaces; $\varepsilon_{gi} = \varphi_{ii}\varepsilon_1 + \varphi_{ik}\varepsilon_1'$, $\varepsilon_{gk} = \varphi_{kk}\varepsilon_1'' + \varphi_{ki}\varepsilon_1'$. The average volume emissivity for the whole enclosure is calculated from the relationship

$$(F_i + F_k)\,\varepsilon_{av} = F_i\varepsilon_{gi} + F_k\varepsilon_{gk}.$$

Finally, the value of N_{gg} can be determined from the equation

$$(F_i + F_k)\,\varepsilon_{av}(1 - N_{gg}) = F_i\varepsilon_{gi}N_{gi} + F_k\varepsilon_{gk}N_{gk}.$$

In the calculation of ε_m, ε_m', ε_m'' and a_m, a_m', a_m'' the same effective beam lengths are used.

A set of special relationships is obtained from the given general formulas. As an example, we write the formula for the emissivity of a cavity obtained by truncation of the enclosure by a plane. The cavity is filled with a nongray medium at the same temperature. The radiation emerges through an opening in k.

$$\varepsilon^* = N_{ki} + N_{kg}, \text{ where } a_m = \varepsilon_m, \ \varphi_{kk} = 0, \ \varphi_{ki} = 1.$$

By algebraic transformation we obtain the result

$$\varepsilon^* = \frac{A_i}{1 - \varphi_{ii}R_i} + R_i\varphi_{ik}(\varepsilon_2^* + \varphi_{ii}R_i\varepsilon_3^* + \varphi_{ii}^2 R_i^2 \varepsilon_4^* + \ldots).$$

We recall that the effective broken path lengths $l_2^*, l_3^*, \ldots$, from which $\varepsilon_2^*, \varepsilon_3^*, \ldots$, are calculated, are traced from the series of angular coefficients $\varphi_{ki}\varphi_{ik}$; $\varphi_{ki}\varphi_{ii}\varphi_{ik}$; .... The coefficient $\varphi_{ki} = 1$ is omitted in the formula.

§ 5. Exact Formulas for Enclosure of Two Flat Gray (1 and 2) and Two Concave Black (3 and 4) Surfaces

A consideration of multiple reflections gives

$$N_{11} = A_1\varphi_{12}\varphi_{21}R_2[(1 - M)^{-1} - (a_2 + Ma_4 + M^2a_6 + \ldots)],$$
$$N_{12} = A_2\varphi_{12}[(1 - M)^{-1} - (a_1 + Ma_3 + M^2a_5 + \ldots)],$$
$$N_{13} = \varphi_{13}[(1 - M)^{-1} - (a_1' + Ma_3' + M^2a_5' + \ldots)] + \varphi_{12}\varphi_{23}R_2[(1 - M)^{-1} - (a_2' + Ma_4' + M^2a_6' + \ldots)].$$

Here $M = \varphi_{12}\varphi_{21}R_1R_2$. The expression for N_{14} is the same as for N_{13} with the substitution $3 \to 4$ and $a_m' \to a_m''$. Finally,

$$N_{1g} = A_1\varphi_{12}\varphi_{21}R_2(a_2 + Ma_4 + \ldots) + A_2\varphi_{12}(a_1 + Ma_3 + \ldots)$$

$$+ \varphi_{13}(a_1' + Ma_3' + \ldots) + \varphi_{12}R_2\varphi_{23}(a_2' + Ma_4' + \ldots) + \varphi_{14}(a_1'' + Ma_3'' + \ldots) + \varphi_{12}R_2\varphi_{24}(a_2'' + Ma_4'' + \ldots).$$

We note that the values of a_m here are determined independently of those in the preceding section. The remaining N_{ik} in this system of bodies are easily found from the reciprocal and closure relations. As a special case we consider a plane-parallel layer: $\varphi_{13} = \varphi_{14} = 0$, $\varphi_{12} = \varphi_{21} = 1$:

$$N_{12} = A_1A_2[1 - a_1 + R_1R_2(1 - a_3) + \ldots],$$

$$N_{11} = A_1^2R_2[1 - a_2 + R_1R_2(1 - a_4) + \ldots],$$

$$N_{1g} = a_1 + R_1R_2(a_3 - a_2) + \ldots + R_2[a_2 - a_1 + R_1R_2(a_4 - a_3) + \ldots].$$

With $R_1 = R_2 = R$ and $T_1 = T_2 = T_g$ we obtain the formula published in [3],

$$AN_{1g} = A\bar{\varepsilon} = A^2(\varepsilon_1 + R\varepsilon_2 + R^2\varepsilon_3 + \ldots). \tag{5}$$

All the derived formulas are suitable for both diffuse and specular reflections. The differences consist in the methods of calculating ε_m and a_m. In the case of diffuse reflection $\varepsilon_m \equiv \varepsilon(ml_{e1})$, $a_m \equiv a(ml_{e1})$. In the case of specular reflection ε_m and a_m are determined for a body with a thickness increased by a factor m in all directions. In this case $l_{e2} < 2l_{e1}$, $l_{e3} < l_{e2} + l_{e1}$, etc. The direct use of the VTI nomograms for l_{e1}, $2l_{e1}$, etc., is valid for diffuse radiation. If there are nomograms or tables of emissivities of a layer, sphere, cylinder, or other bodies, they can be used in a similar way for specular reflection.

§ 6. Approximate Formulas

The exact solutions given are infinite series. Their convergence for high reflection coefficients is slow. We consider approximate formulas obtained by exact calculation of the first few terms of the series and curtailment of the rest. We illustrate the method by the example of a typical and simple formula for the reduced emissivity of a volume in the case of heat exchange with a gray enclosure. It follows from (5) that

$$\bar{\varepsilon} = A(\varepsilon_1 + R\varepsilon_2 + R^2\varepsilon_3 + \ldots).$$

This can be converted to

$$\bar{\varepsilon} = \varepsilon_1 + R(\varepsilon_2 - \varepsilon_1) + R^2(\varepsilon_3 - \varepsilon_2) + \ldots \tag{6}$$

We can now rewrite the formula, using the transmittances

$$D_{jm} = (\varepsilon_j - \varepsilon_{j-m})/\varepsilon_m \quad (j > m), \tag{7}$$

where m and j are the number of passages of the beams through the volume due to multiple reflections. Here ε_m determines the spectrum of the beam for which on a path consisting of (j – m) parts the transmittance D_{jm} is defined.

For m = 1

$$D_{21} = (\varepsilon_2 - \varepsilon_1)/\varepsilon_1; \quad D_{31} = (\varepsilon_3 - \varepsilon_2)/\varepsilon_1; \ \ldots .$$

Substitution in (6) gives

$$\bar{\varepsilon} = \varepsilon_1(1 + RD_{21} + R^2D_{31} + R^3D_{41} + \ldots).$$

For m = 2

$$D_{32} = (\varepsilon_3 - \varepsilon_1)/\varepsilon_2; \quad D_{42} = (\varepsilon_4 - \varepsilon_2)/\varepsilon_2; \ \ldots .$$

Substitution in (6) and transformations give

$$(1 + R)\bar{\varepsilon} = A\varepsilon_1 + R\varepsilon_2(1 + RD_{32} + R^2D_{42} + \ldots).$$

Here and henceforth we use a typical transformation

$$\varepsilon_j - \varepsilon_{j-1} = \varepsilon_m \frac{\varepsilon_j - \varepsilon_{j-m} - (\varepsilon_{j-1} - \varepsilon_{j-m})}{\varepsilon_m} = \varepsilon_m D_{jm} - \varepsilon_{m-1}D_{(j-1)(m-1)}.$$

For any m

$$(1 + R + \ldots R^{m-1})\bar{\varepsilon} = \varepsilon_1 + R\varepsilon_2 + \ldots R^{m-2}\varepsilon_{m-1} + R^{m-1}\varepsilon_m(1 + RD_{m+1,m} + R^2D_{m+2,m} + \ldots).$$

The first approximation consists in using the equation $D_{jm} = D_{m+1,m}^{j-m}$, which is valid only in the case of an exponential absorption law. In this case the "tail" of the series is reduced to

$$1 + RD_{m+1,m} + R^2D_{m+2,m} + \ldots = (1 - RD_{m+1,m})^{-1}.$$

The result is an underestimate. The second approximation consists in using the equation

$$D'_{jm} = (D_{m+2,m}/D_{m+1,m})^{j-m}.$$

The ratio in the parentheses has the sense of the transmittance of the second region after the m-th region. The result is an overestimate to approximately the same extent. Hence, we take the geometric mean $D''_{jm} = \sqrt{DD'} = (\sqrt{D_{m+2,m}})^{j-m}$. Finally,

$$(1 + R + \ldots R^{m-1})\bar{\varepsilon} = \varepsilon_1 + R\varepsilon_2 + \ldots + R^{m-2}\varepsilon_{m-1} + R^{m-1}\varepsilon_m (1 - R\sqrt{D_{m+2,m}})^{-1}, \tag{8}$$

where $D_{m+2,m} = (\varepsilon_{m+2} - \varepsilon_2)/\varepsilon_m$.

We will show that the derived formulas can be curtailed by the same method. The formulas contain the typical series

$$a_1 + \varphi_{ii}R_i a_2 + \varphi_{ii}^2 R_i^2 a_3 + \ldots = (1 - \varphi_{ii}R_i)^{-1}[a_1 + \varphi_{ii}R_i(a_2 - a_1) + \varphi_{ii}^2 R_i^2 (a_3 - a_2) + \ldots].$$

The parentheses contain an expression similar to (6). Instead of (7) we use

$$D_{jm} = \frac{a_j - a_{j-m}}{a_m} \quad (j > m).$$

The rest of the procedure is repeated.

§ 7. Numerical Calculations

To assess the method of curtailment of the series we need to test the typical formula (8). When m = 1 it assumes a very simple form

$$\bar{\varepsilon} = \frac{\varepsilon_1}{1 - R\sqrt{D_{31}}}, \quad \text{where } D_{31} = (\varepsilon_3 - \varepsilon_2)/\varepsilon_1. \tag{9}$$

With increase in m the error of formula (8) decreases, and when $m \to \infty$ the formula becomes exact. Table 1 gives: $\bar{\varepsilon}_N$ is the reduced emissivity of a plane-parallel layer for specular reflection; the figures were

TABLE 1. Comparison of Reduced Emissivities of Volume of Gas in Gray Enclosure with Reflection Coefficient R; $\bar{\varepsilon}_N$, $\bar{\varepsilon}$ are from Exact Formulas; $\bar{\varepsilon}_a$ is from Approximate Formula (9); $\bar{\varepsilon}_s$ is from Formula in [8]

Conditions	R	$\bar{\varepsilon}_H$	$\bar{\varepsilon}_a$	$\bar{\varepsilon}_s$
CO_2 layer. Thickness x_0 = 0.05 m-atm, t = 1000 °C. Specular reflection	0.2	0.1032	0.106	0.1084
	0.4	0.1109	0.116	0.1220
	0.6	0.1229	0.128	0.1394
	0.8	0.1469	0.143	0.1627
H_2O layer. Thickness x_0 = 0.05 m-atm, t = 1000 °C. Specular reflection	0.2	0.1047	0.106	0.1043
	0.4	0.1201	0.123	0.1174
	0.6	0.1450	0.146	0.1341
	0.8	0.1948	0.178	0.1565
Conditions	**R**	$\bar{\varepsilon}$	$\bar{\varepsilon}_a$	$\bar{\varepsilon}_s$
CO_2 volume. x_e = 0.001 m-atm, t = 800 °C. Diffuse reflection	0.2	0.0140	0.0140	0.0133
	0.4	0.0170	0.0169	0.0150
	0.6	0.0217	0.0213	0.0171
	0.8	0.0310	0.0286	0.0200
H_2O volume. x_e = 0.01 m-atm, t = 800 °C. Diffuse reflection	0.2	0.0281	0.0284	0.0267
	0.4	0.0344	0.0362	0.0300
	0.6	0.0454	0.0447	0.0343
	0.8	0.0700	0.0628	0.0400

obtained and kindly provided by A.S.Nevskii; $\bar{\varepsilon}$ is the reduced emissivity of a volume with effective thickness x_e for diffuse reflection; $\bar{\varepsilon}_a$ is the approximate value obtained from formula (9). With R = 0.8 we took 20 terms of the series, but the remainder was still substantial, according to the estimate. In these conditions the simple formula (9) gave a very good result. In [7] we obtained the best approximation of the same form as (9): $\bar{\varepsilon} = \varepsilon_1(1 - CR)^{-1}$. The choice of the value of C, however, was not so good as here. Of the other approximate formulas we give those proposed by Splett in [8] after a review, including the review in [3]: $\bar{\varepsilon}_s = 2\varepsilon_1(2 - R)^{-1}$. Table 1 shows that formula (9) is much better.

NOTATION

$x = \int p\,dl$	is the beam path reduced according to partial pressure p;
$\varepsilon(x, T)$	is the emissivity of gas on region x;
$a(x, T_g, T)$	is the absorptivity of gas for black beam with temperature T;
$\tilde{\Phi}, \tilde{\varphi}_*$	are the square matrices, explained in § 2;
A	is the absorption coefficient;
R	is the reflection coefficient (also scattering coefficient, in § 2);
Q_{ab}, Q_o, Q_r	are the absorbed, original, and resultant fluxes of zones;
A, R	are the diagonal matrices, in § 2;
Q_{ab}, Q_o, Q_r	are the columns;
N	is the dimensionless "resolving" angular coefficient, meaning given in § 2;
F	is the surface of zone;
ψ_{ik}	is the generalized angular coefficient for surfaces F_i and F_k;
φ_{ik}	is the generalized angular coefficient for diathermal medium;
W_{in}, W	explained in § 3;
D_m, a_m, ε_m	are the transmittance, absorptivity, and emissivity for broken path consisting of m regions; dashes indicate differences in broken paths;
l_e	is the effective beam length;
ε_{gi}	is the emissivity;
ε^*	is the emissivity of gray enclosure filled with nongray gas;
$\bar{\varepsilon}$	is the reduced emissivity of volume for heat exchange with enclosure;
D_{jm}	is the transmittance of j – m regions for beam emitted by m regions in series.

Subscripts

g denotes the volume, the first subscript indicates the source.

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